Many of the designs found on ancient murals and pottery derive their beauty from complex patterns of geometric shapes. Modern sculptures, buildings, and bridges also rely on geometric characteristics for beauty and durability. Today, computers give graphic designers, architects, and engineers a means for experimenting with design elements.

- **Jewelers** (page 159) design jewelry, cut gems, make repairs, and use their understanding of geometry to appraise gems. Jewelers need skills in art, math, mechanical drawing, and chemistry to practice their trade.

- **Animators** (page 177) create pictures that are filmed frame by frame to create motion. Many animators use computers to create three-dimensional backgrounds and characters. Animators need an understanding of perspective to create realistic drawings.
**Suspension Bridges of New York**

<table>
<thead>
<tr>
<th>Name</th>
<th>Year opened</th>
<th>Length of main span</th>
<th>Height of towers</th>
<th>Clearance above water</th>
<th>Cost of original structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brooklyn Bridge</td>
<td>1883</td>
<td>1595.5 ft</td>
<td>276.5 ft</td>
<td>135 ft</td>
<td>$15,100,000</td>
</tr>
<tr>
<td>Williamsburg Bridge</td>
<td>1903</td>
<td>1600 ft</td>
<td>310 ft</td>
<td>135 ft</td>
<td>$30,000,000</td>
</tr>
<tr>
<td>Manhattan Bridge</td>
<td>1909</td>
<td>1470 ft</td>
<td>322 ft</td>
<td>135 ft</td>
<td>$25,000,000</td>
</tr>
<tr>
<td>George Washington Bridge</td>
<td>1931</td>
<td>3500 ft</td>
<td>604 ft</td>
<td>213 ft</td>
<td>$59,000,000</td>
</tr>
<tr>
<td>Verrazano Narrows Bridge</td>
<td>1962*</td>
<td>4260 ft</td>
<td>693 ft</td>
<td>228 ft</td>
<td>$320,126,000</td>
</tr>
<tr>
<td>Verrazano Narrows Bridge</td>
<td>1969*</td>
<td>4260 ft</td>
<td>693 ft</td>
<td>228 ft</td>
<td>$320,126,000</td>
</tr>
</tbody>
</table>

* lower deck added

**Data Activity: Suspension Bridges of New York**

Use the table for Questions 1–4.

1. For which bridge is the ratio of tower height to length of main span closest to 1 : 5?

2. The width of the George Washington Bridge is 119 ft. What is the area of the main span in square feet? *Hint: Use the formula \( A = \ell w \), where \( \ell \) = length and \( w \) = width.*

3. Suppose you want to make a scale model of the Verrazano Narrows Bridge. The entire model must be no more than 4 ft in length. Choose a scale and then find the model’s length of main span, clearance above water, and height of towers.

4. By what percent did the cost of building a bridge increase from 1931 to 1964? Round to the nearest whole percent. Disregard differences in bridge size.

**CHAPTER INVESTIGATION**

A truss bridge covers the span between two supports using a system of angular braces to support weight. Triangles are often used in the building of truss bridges because the triangle is the strongest supporting polygon. The railroads often built truss bridges to support the weight of heavy locomotives.

**Working Together**

Design a system for a truss bridge similar to the examples shown throughout this chapter. Carefully label the measurements and angles in your design. Build a model of the truss using straws, toothpicks or other suitable materials. Use the Chapter Investigation icons to assist your group in designing the structure.
The skills on these two pages are ones you have already learned. Stretch your memory and complete the exercises. For additional practice on these and more prerequisite skills, see pages 654–661.

You will be learning more about geometric shapes and their properties. Now is a good time to review what you already have learned about polygons and triangles.

**POLYGONS**

A **polygon** is a closed plane figure formed by joining 3 or more line segments at their endpoints. Polygons are named for the number of their sides.

Tell whether each figure is a polygon. If not, give a reason.

1. 2. 3. 4.

Give the best name for each polygon.

5. 6. 7. 8.

9. 10. 11. 12.

**CONGRUENT TRIANGLES**

Triangles can be determined to be congruent, or having the same size and shape, by three tests:

- Triangles with the same measure of two angles and the included side are congruent.
- Triangles with the same measure of two sides and the included angle are congruent.
- Triangles with the same measures of three sides are congruent.
Tell whether each pair of triangles is congruent. If they are congruent, identify how you determined congruency.

13.  
14.  
15.  

16.  
17.  
18.  

**ANGLES OF TRIANGLES**

**Example**  The sum of the measures of the interior angles of any triangle is $180^\circ$. Find the unknown measure.

\[53^\circ + 73^\circ + x^\circ = 180^\circ\]
\[126^\circ + x^\circ = 180^\circ\]
\[x^\circ = 54^\circ\]

Find the unknown measures in each triangle.

19.  
20.  
21.  

22.  
23.  
24.  

25.  
26.  
27.  

Chapter 4  Are You Ready?
Work with a partner.

1. Using a pencil and a straightedge, draw and label a triangle as shown below. Carefully cut on the straight lines. Then tear off the four labeled angles.

   ![](image)

   ⇒

   a. What relationships can you discover among the four angles?
   b. Using these relationships, make at least two conjectures that you think apply to all triangles.

BUILD UNDERSTANDING

A triangle is the figure formed by the segments that join three noncollinear points. Each segment is called a side of the triangle. Each point is called a vertex (plural: vertices). The angles determined by the sides are called the interior angles of the triangle.

![Diagram of triangle ABC](image)

- sides: \( \overline{AB} \), \( \overline{BC} \), and \( \overline{AC} \)
- vertices: points A, B, and C
- interior angles: \( \angle A \), \( \angle B \), and \( \angle C \)

Often a triangle is classified by relationships among its sides.

- Equilateral triangle: three sides of equal length
- Isosceles triangle: at least two sides of equal length
- Scalene triangle: no sides of equal length

A triangle also can be classified by its angles.

- Acute triangle: three acute angles
- Obtuse triangle: one obtuse angle
- Right triangle: one right angle
- Equiangular triangle: three angles equal in measure

You probably remember a special property of the measures of the angles of a triangle. Since the fact is a theorem, it can be proved true.

**The Triangle-Sum Theorem**

The sum of the measures of the angles of a triangle is 180°.
As you read the proof of the theorem, notice that it makes use of a line that intersects one of the vertices of the triangle and is parallel to the opposite side. This line, which has been added to the diagram to help in the proof, is called an auxiliary line.

Given $\triangle ABC; \overline{DB} \parallel \overline{AC}$
Prove $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\triangle ABC; \overline{DB} \parallel \overline{AC}$</td>
<td>1. given</td>
</tr>
<tr>
<td>2. $m\angle 4 + m\angle 2 = m\angle DBC$</td>
<td>2. angle addition postulate</td>
</tr>
<tr>
<td>$m\angle DBC + m\angle 5 = 180^\circ$</td>
<td>3. substitution property</td>
</tr>
<tr>
<td>3. $m\angle 4 + m\angle 2 + m\angle 5 = 180^\circ$</td>
<td>4. If two parallel lines are cut by a transversal, then alternate interior angles are equal in measure.</td>
</tr>
<tr>
<td>4. $m\angle 4 = m\angle 1$</td>
<td></td>
</tr>
<tr>
<td>$m\angle 5 = m\angle 3$</td>
<td></td>
</tr>
<tr>
<td>5. $m\angle 1 + m\angle 2 + m\angle 3 = 180^\circ$</td>
<td>5. substitution property</td>
</tr>
</tbody>
</table>

The triangle-sum theorem is useful in art and design.

**Example 1**

**TECHNICAL ART** An artist is using the figure at the right to create a diagram for a publication. Using the triangle-sum theorem, find $m\angle Q$.

**Solution**

From the triangle-sum theorem, you know that the sum of the measures of the angles of a triangle is $180^\circ$. Use this fact to write and solve an equation.

$$m\angle P + m\angle Q + m\angle R = 180$$

$$27 + (g + 9) + 2g = 180$$  Combine like terms.

$$3g + 36 = 180$$  Add $-36$ to each side.

$$3g = 144$$  Multiply each side by $\frac{1}{3}$.

$$g = 48$$

So, the value of $g$ is 48. From the figure $m\angle Q = (g + 9)^\circ$.

Substituting 48 for $g$, $m\angle Q = (48 + 9)^\circ = 57^\circ$.

An exterior angle of a triangle is an angle that is both adjacent to and supplementary to an interior angle, as shown at the right. The following is an important theorem concerning exterior angles.

**The Exterior Angle Theorem**

The measure of an exterior angle of a triangle is equal to the sum of the measures of the two nonadjacent (remote) interior angles.
You will have an opportunity to prove the exterior angle theorem in Exercise 12 on the following page. Example 2 shows one way the theorem can be applied.

**Example 2**

In the figure at the right, find $m\angle EFG$.

**Solution**

Notice that $\angle DEG$ is an exterior angle, while $\angle EGF$ and $\angle EFG$ are nonadjacent interior angles. Use the exterior angle theorem to write and solve an equation.

\[
m\angle DEG = m\angle EGF + m\angle EFG
\]

\[
115 = 6z + (3z - 2)
\]

Combine like terms.

\[
115 = 9z - 2
\]

Add 2 to each side.

\[
117 = 9z
\]

Multiply each side by $\frac{1}{9}$.

\[
13 = z
\]

So, the value of $z$ is 13. From the figure, $m\angle EFG = (3z - 2)^\circ$. Substituting 13 for $z$, $m\angle EFG = (3 \cdot 13 - 2)^\circ = (39 - 2)^\circ = 37^\circ$.

**Try These Exercises**

Refer to $\triangle RST$ below. Find the measure of each angle.

1. $\angle R$
2. $\angle S$
3. $\angle T$

Refer to $\triangle XYZ$ below. Find the measure of each angle.

4. $\angle YXZ$
5. $\angle XZW$
6. $\angle XZY$

**Practice Exercises** • For Extra Practice, see page 673.

Find the value of $x$ in each figure.

7.

8.

9.

10. **WRITING MATH** How many exterior angles does a triangle have? Draw a triangle and label all its exterior angles.

11. The measure of the largest angle of a triangle is twice the measure of the smallest angle. The measure of the third angle is $10^\circ$ less than the measure of the largest angle. Find all three measures.
12. In the figure below, \( XZ \perp YZ \). Find \( m\angle XYZ \).

13. **BRIDGE BUILDING** The plans for a bridge call for the addition of triangular bracing to increase the amount of weight the bridge can hold. On the plans, \( \triangle FGH \) is drawn so that the \( m\angle F \) is 14° less than three times the \( m\angle G \), and \( \angle H \) is a right angle. Find the measure of each angle.

**GEOMETRY SOFTWARE** On a coordinate plane, draw the triangle with the given vertices. Measure all sides and angles. Then classify the triangle, first by its sides, then by its angles.

14. \( A(-5, 0); B(1, 2); C(1, -2) \)
15. \( J(-1, -3); K(6, 2); L(-7, 1) \)
16. \( R(1, -5); S(-3, -1); T(6, 0) \)
17. \( D(3, -5); E(-4, -3); F(-2, 4) \)

18. Copy and complete this proof of the exterior angle theorem.

**Given** \( \triangle ABC \), with exterior \( \angle 4 \)

**Prove** \( \angle 1 \equiv \angle 3 \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \angle 1 \equiv \angle 3 )</td>
<td>1. <strong>?</strong></td>
</tr>
<tr>
<td>2. ( m\angle 1 + m\angle 2 + m\angle 3 = 180° )</td>
<td>2. <strong>?</strong></td>
</tr>
<tr>
<td>3. ( m\angle 1 + m\angle 2 + m\angle 3 = 180° )</td>
<td>3. <strong>?</strong></td>
</tr>
<tr>
<td>4. ( m\angle 1 + m\angle 2 + m\angle 3 = m\angle 4 + m\angle 3 )</td>
<td>4. <strong>?</strong></td>
</tr>
<tr>
<td>5. <strong>?</strong></td>
<td>5. <strong>?</strong></td>
</tr>
</tbody>
</table>

**EXTENDED PRACTICE EXERCISES**

Determine whether each statement is **always**, **sometimes**, or **never** true.

19. Two interior angles of a triangle are obtuse angles.
20. Two interior angles of a triangle are acute angles.
21. An exterior angle of a triangle is an obtuse angle.
22. The measure of an exterior angle of a triangle is greater than the measure of each nonadjacent interior angle.

**MIXED REVIEW EXERCISES**

Find each length. (Lesson 3-1)

23. In the figure below, \( AC = 75 \).
   Find \( BC \).

24. In the figure below, \( PR = 138 \).
   Find \( PQ \).

Find the mean, median and mode of each set of data. (Lesson 2-7)

25. \( 4 \ 8 \ 7 \ 10 \ 8 \ 8 \ 3 \)
   \( 7 \ 9 \ 14 \ 3 \ 5 \)
26. \( 8 \ 7 \ 3 \ 9 \ 9 \ 5 \ 7 \ 9 \)
   \( 1 \ 3 \ 2 \ 6 \ 9 \ 1 \ 4 \)
For the following activity, use a protractor and a metric ruler, or use geometry software. Give lengths to the nearest tenth of a centimeter, and give angle measures to the nearest degree.

a. Draw \(\triangle ABC\), with \(\angle A = 40^\circ\), \(AB = 7\) cm, and \(\angle B = 60^\circ\). What is the measure of \(\angle C\)? What is the length of \(AC\) of \(BC\)?

b. Draw \(\triangle DEF\), with \(DF = 5\) cm, \(\angle D = 120^\circ\), and \(DE = 6\) cm. What is the measure of \(\angle E\) of \(\angle F\)? What is the length of \(EF\)?

c. Draw \(\triangle GHJ\), with \(\angle G = 35^\circ\), \(\angle H = 45^\circ\), and \(\angle J = 100^\circ\). What is the length of \(GH\) of \(GJ\) or \(HJ\)?

d. Draw \(\triangle KLM\), with \(KM = 3\) cm, \(KL = 6\) cm, and \(LM = 4\) cm. What is the measure of \(\angle K\) of \(\angle L\) of \(\angle M\)?

When two geometric figures have the same size and shape, they are said to be congruent. The symbol for congruence is \(\cong\).

It is fairly easy to recognize when segments and angles are congruent. Congruent segments are segments with the same length. Congruent angles are angles with the same measure.

**Congruent triangles** are two triangles whose vertices can be paired in such a way that each angle and side of one triangle is congruent to a corresponding angle or corresponding side of the other. For instance, the markings in the triangles at the right indicate these six congruences.

\[
\begin{align*}
\angle A & \cong \angle Z \\
\angle B & \cong \angle X \\
\angle C & \cong \angle Y \\
\overline{AB} & \cong \overline{ZX} \\
\overline{BC} & \cong \overline{XY} \\
\overline{AC} & \cong \overline{ZY}
\end{align*}
\]

So, the triangles are congruent, and you can pair the vertices in the following correspondence.

\[
\begin{align*}
A & \leftrightarrow Z \\
B & \leftrightarrow X \\
C & \leftrightarrow Y
\end{align*}
\]

To state the congruence between the triangles, you list the vertices of each triangle in the same order as this correspondence.
You can use given information to prove that two triangles are congruent. One way to do this is to show the triangles are congruent by definition. That is, you prove that all six parts of one triangle are congruent to six corresponding parts of the other. However, this can be quite cumbersome.

Fortunately, triangles have special properties that allow you to prove triangles congruent by identifying only three sets of corresponding parts. The first way to do this is summarized in the SSS postulate.

**Postulate 11**

**The SSS Postulate** If three sides of one triangle are congruent to three sides of another triangle, then the triangles are congruent.

### Example 1

**ANIMATION** The figure shown is part of a perspective drawing for a background scene of a city. How can the artist be sure that the two triangles are congruent?

**Given**  
$JK \cong JM; KL \cong ML$

**Prove** $\triangle JKL \cong \triangle JML$

**Solution**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $JK \cong JM; KL \cong ML$</td>
<td>1. given</td>
</tr>
<tr>
<td>2. $JL \cong JL$</td>
<td>2. reflexive property</td>
</tr>
<tr>
<td>3. $\triangle JKL \cong \triangle JML$</td>
<td>3. SSS postulate</td>
</tr>
</tbody>
</table>

**GEOMETRY SOFTWARE** Use geometry software to explore the postulate. Draw two triangles so that the three sides of one triangle are congruent to the three corresponding sides of the other triangle. Measure the interior angles of both triangles. They are also congruent.

Sometimes it is helpful to describe the parts of a triangle in terms of their relative positions.

Each angle of a triangle is formed by two sides of the triangle. In relation to the two sides, this angle is called the included angle. Each side of a triangle is included in two angles of the triangle. In relation to the two angles, this side is called the included side.

Using these terms, it is now possible to describe two additional ways of showing that two triangles are congruent.

**Postulate 12**

**The SAS Postulate** If two sides and the included angle of one triangle are congruent to two corresponding sides and the included angle of another triangle, then the triangles are congruent.

**Postulate 13**

**The ASA Postulate** If two angles and the included side of one triangle are congruent to two corresponding angles and the included side of another triangle, then the triangles are congruent.
Example 2

Given \( \overline{VW} \equiv \overline{ZY}; \angle V \equiv \angle Z \)
\( \overline{VW} \perp \overline{WY}; \overline{ZY} \perp \overline{WY} \)

Prove \( \triangle VWX \equiv \triangle ZYX \)

Solution

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{VW} \equiv \overline{ZY}; \angle V \equiv \angle Z ) ( \overline{VW} \perp \overline{WY}; \overline{ZY} \perp \overline{WY} )</td>
<td>1. given</td>
</tr>
<tr>
<td>2. ( \angle W ) and ( \angle Y ) are right angles.</td>
<td>2. definition of perpendicular lines</td>
</tr>
<tr>
<td>3. ( m\angle W = 90^\circ; m\angle Y = 90^\circ )</td>
<td>3. definition of right angle</td>
</tr>
<tr>
<td>4. ( m\angle W = m\angle Y, or \angle W \equiv \angle Y )</td>
<td>4. transitive property of equality</td>
</tr>
<tr>
<td>5. ( \triangle VWX \equiv \triangle ZYX )</td>
<td>5. ASA postulate</td>
</tr>
</tbody>
</table>

Try These Exercises

1. Copy and complete this proof.

   Given \( \overline{RQ} \equiv \overline{RS}; \overline{RT} \) bisects \( \angle QRS \).

   Prove \( \triangle QRT \equiv \triangle SRT \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. <em><strong>?</strong></em></td>
<td>1. <em><strong>?</strong></em></td>
</tr>
<tr>
<td>2. ( m\angle 1 = m\angle 2, or \angle 1 \equiv \angle 2 )</td>
<td>2. definition of <em><strong>?</strong></em></td>
</tr>
<tr>
<td>3. <em><strong>?</strong></em></td>
<td>3. <em><strong>?</strong></em> property</td>
</tr>
<tr>
<td>4. ( \triangle QRT \equiv \triangle SRT )</td>
<td>4. <em><strong>?</strong></em></td>
</tr>
</tbody>
</table>

2. CONSTRUCTION Plans call for triangular bracing to be added to a horizontal beam. Prove the triangles are congruent by writing a two-column proof.

   Given \( \overline{GL} \equiv \overline{JK}; \overline{HL} \equiv \overline{HK} \)

   Point \( H \) is the midpoint of \( \overline{GJ} \).

   Prove \( \triangle GHL \equiv \triangle JHK \)

Practice Exercises • For Extra Practice, see page 673.

3. Write a two-column proof.

   Given \( \overline{AB} \equiv \overline{CB}; \overline{EB} \equiv \overline{DB} \)

   \( AD \) and \( CE \) intersect at point \( B \).

   Prove \( \triangle ABE \equiv \triangle CBD \)
**ENGINEERING** The figures in Exercises 4–7 are taken from truss bridge designs. Each figure contains a pair of congruent overlapping triangles.

Use the given information to complete the congruence statement. Then name the postulate that would be used to prove the congruence. (You do not need to write the proof.)

4. **Given** \[PQ \cong SR; QS \cong RP\]

\[\triangle PQS \cong \ ? \]

5. **Given** \[AC \cong EC; \angle A \cong \angle E\]

\[\triangle ACD \cong \ ? \]

6. **Given** \[EH \cong FG; EH \perp EF; FG \perp EF\]

\[\triangle HEF \cong \ ? \]

7. **Given** \[JL \cong MK; \angle J \cong \angle M; JN \cong MN\]

\[\triangle JNL \cong \ ? \]

**GEOMETRY SOFTWARE** Use geometry software or paper and pencil to draw the figures in Exercises 8–9 on a coordinate plane.

8. Draw \(\triangle MNP\) with vertices \(M(-5, 5), N(3, 5),\) and \(P(3, -6)\). Then draw \(\triangle QRS\) with vertices \(Q(-4, 2), R(7, -6)\) and \(S(-4, -6)\). Explain how you know that the triangles are congruent. Then state the congruence.

9. Draw \(\triangle ABC\) with vertices \(A(-3, 5), B(6, 5),\) and \(C(6, -8)\). Then graph points \(X(2, 2)\) and \(Y(-7, 2)\). Find two possible coordinates of a point \(Z\) so that \(\triangle ABC \cong \triangle XYZ\).

**10. CHAPTER INVESTIGATION** Design a 20-foot side section of a truss bridge. Draw your design to the scale 1 in. = 2 ft.

**EXTENDED PRACTICE EXERCISES**

11. Write a proof of the following statement.

   If two legs of one right triangle are congruent to two legs of another right triangle, then the triangles are congruent.

12. **WRITING MATH** Write a convincing argument to explain why there is no SSA postulate for congruence of triangles.

**MIXED REVIEW EXERCISES**

Find the measure of each angle. (Lesson 3-2)

13. \(\angle ABD\)  
14. \(\angle CBD\)  
15. \(\angle EFH\)  
16. \(\angle GFH\)

\[\angle ABD = \frac{4x + 8}{(2x - 3)}\]  
\[\angle CBD = \frac{3x + 4}{(3x - 2)}\]  
\[\angle EFH = \frac{(4x)}{D}\]  
\[\angle GFH = \frac{E}{H}\]
Review and Practice Your Skills

**PRACTICE ▶ Lesson 4-1**

Find the value of $x$ in each figure.

1. \[ \triangle \text{with angles } 28^\circ, 57^\circ \]

2. \[ \triangle \text{with angles } 132^\circ, x^\circ \]

3. \[ \triangle \text{with angles } 86^\circ, 63^\circ, x^\circ \]

4. \[ \triangle \text{with angles } x^\circ, 2x^\circ \]

5. \[ \triangle \text{with angles } 65^\circ, 75^\circ, x^\circ \]

6. \[ \triangle \text{with angles } (3x)^\circ, (x + 15)^\circ, x^\circ \]

Determine whether each statement is **true** or **false**.

7. If two angles in a triangle are acute, then the third angle is always obtuse.

8. If one angle in a triangle is obtuse, then the other two angles are always acute.

9. If one exterior angle of a triangle is obtuse, then all three interior angles are acute.

10. If two angles in a triangle are congruent, then the triangle is equiangular.

On a coordinate plane, sketch the triangle with the given vertices. Then classify the triangle, first by its sides, then by its angles.

11. \( A(2, 2); B(-3, -3); C(-3, 2) \)

12. \( X(6, -2); Y(-4, -2); Z(1, 0) \)

13. \( M(-1, 4); N(1, 0); P(-4, 0) \)

**PRACTICE ▶ Lesson 4-2**

14. Copy and complete this proof.

**Given** \( AB \parallel DE; C \text{ is the midpoint of } BD \).

**Prove** \( \triangle ABC \cong \triangle EDC \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( AB \parallel DE )</td>
<td>1. _<strong>?</strong></td>
</tr>
<tr>
<td>2. ( \angle B \equiv \angle D )</td>
<td>2. _<strong>?</strong></td>
</tr>
<tr>
<td>3. ( \angle BCA \equiv \angle ECD )</td>
<td>3. _<strong>?</strong></td>
</tr>
<tr>
<td>4. _<strong>?</strong></td>
<td>4. given</td>
</tr>
<tr>
<td>5. ( BC \equiv CD )</td>
<td>5. _<strong>?</strong></td>
</tr>
<tr>
<td>6. _<strong>?</strong></td>
<td>6. ASA postulate</td>
</tr>
</tbody>
</table>

15. Name all the pairs of congruent parts in these triangles. Then state the congruence between the triangles.

16. **True or false**: If three angles of one triangle are congruent to three angles of another triangle, then the triangles are congruent.
Determine whether each statement is always, sometimes, or never true. (Lesson 4-1)

17. There are two exterior angles at each vertex of a triangle.
18. An exterior angle of a triangle is an acute angle.
19. Two of the three angles in a triangle are complementary angles.
20. The sum of the measures of the angles in a triangle is 90°.

Use the given information to complete each congruence statement. Then name the postulate that would be used to prove the congruence. (You do not need to write the proof.) (Lesson 4-2)

21. Given \(PQ \cong NO; QR \cong MO; \angle Q \cong \angle O\) \(\triangle PQR \cong ?\)

22. Given \(\overline{EF} \cong \overline{EF}; \overline{DF} \cong \overline{LF}; \overline{DE} \cong \overline{LE}\) \(\triangle DYE \cong ?\)

---

**MathWorks Career – Jeweler**

A jeweler designs and repairs jewelry, cuts gems and appraises the value of gemstones and jewelry. Most jewelers go through an apprenticeship program where they work under an experienced jeweler to hone their skills and learn new techniques. A background in art, math, mechanical drawing and chemistry are all useful when working with gems and precious metals. Math skills help a jeweler in the areas of design and gem cutting. Jewelers use computer-aided design (CAD) programs to design jewelry to meet a customer’s expectations. A symmetrically cut gem is a valuable gem. A poorly cut gem becomes a wasted investment for the jeweler.

In the gem cut shown to the right, all triangles shown can be classified as isosceles triangles.

1. What additional classifications can be given to triangle \(ABC\)?
2. What is the measure of \(\angle BCE\)?
3. Sides \(CE\) and \(DE\) are congruent and \(\angle BCE\) and \(\angle EDF\) are congruent. Angle \(DEF\) measures 38°. Are triangles \(BCE\) and \(FDE\) congruent? If so, what postulate could be used to prove the congruence?
Fold a piece of paper and draw a segment on it as shown. Now cut both thicknesses of paper along the segment. Unfold and label the triangle.

1. Are there any perpendicular segments on the triangle?
2. Does any segment lie on an angle bisector of the triangle?
3. List as many congruences as you can among the segments, angles, and triangles that you see on the folded triangle.

**BUILD UNDERSTANDING**

The SSS, SAS, and ASA postulates help you determine a congruence between two triangles by identifying just three pairs of corresponding parts. Once you establish a congruence, you may conclude that all pairs of corresponding parts are congruent. Example 1 shows how this fact can be used to show that two angles are congruent.

**Example 1**

Given \( \overline{AB} \cong \overline{CB} \); \( \overline{AD} \cong \overline{CD} \)

Prove \( \angle A \cong \angle C \)

**Solution**

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{AB} \cong \overline{CB} ); ( \overline{AD} \cong \overline{CD} )</td>
<td>1. given</td>
</tr>
<tr>
<td>2. ( \overline{BD} \cong \overline{BD} )</td>
<td>2. reflexive property</td>
</tr>
<tr>
<td>3. ( \triangle ABD \cong \triangle CBD )</td>
<td>3. SSS postulate</td>
</tr>
<tr>
<td>4. ( \angle A \cong \angle C )</td>
<td>4. Corresponding parts of congruent triangles are congruent.</td>
</tr>
</tbody>
</table>

Corresponding parts of congruent triangles are used in the proofs of many theorems. For example, an isosceles triangle is a triangle with two legs of equal length. The third side is the base. The angles at the base are called the base angles, and the third angle is the vertex angle. CPCTC can be used to prove the following theorem about base angles.

**Reading Math**

The final reason of the proof in Example 1 is *Corresponding parts of congruent triangles are congruent*. This fact is used so often that it is commonly abbreviated CPCTC.
The Isosceles Triangle Theorem

If two sides of a triangle are congruent, then the angles opposite those sides are congruent. This is sometimes stated:

*Base angles of an isosceles triangle are congruent.*

Given \( \triangle ABC \) is isosceles, with base \( \overline{AC} \).

\( \overline{BX} \) bisects \( \angle ABC \).

Prove \( \angle A \cong \angle C \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \triangle ABC ) is isosceles, with base ( \overline{AC} ). ( \overline{BX} ) bisects ( \angle ABC ).</td>
<td>1. given</td>
</tr>
<tr>
<td>2. ( AB \cong CB )</td>
<td>2. definition of isosceles ( \triangle )</td>
</tr>
<tr>
<td>3. ( \angle 1 \cong \angle 2 )</td>
<td>3. definition of ( \angle ) bisector</td>
</tr>
<tr>
<td>4. ( \overline{BX} \cong \overline{BX} )</td>
<td>4. reflexive property</td>
</tr>
<tr>
<td>5. ( \triangle AXB \cong \triangle CXB )</td>
<td>5. SAS postulate</td>
</tr>
<tr>
<td>6. ( \angle A \cong \angle C )</td>
<td>6. CPCTC</td>
</tr>
</tbody>
</table>

**Example 2**

**DESIGN** An artist is positioning the design elements for a new company logo. At the center of the logo is the triangle shown in the figure. Find \( m\angle P \).

**Solution**

Since \( PQ = PR \), \( \triangle PQR \) is isosceles with base \( \overline{QR} \). By the isosceles triangle theorem, \( m\angle R = m\angle Q = 66^\circ \). By the triangle-sum theorem, \( m\angle P + 66^\circ + 66^\circ = 180^\circ \), or \( m\angle P = 48^\circ \).

A statement that follows directly from a theorem is called a [corollary](#). The following are corollaries to the isosceles triangle theorem.

<table>
<thead>
<tr>
<th>Corollary 1</th>
<th>If a triangle is equilateral, then it is equiangular.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Corollary 2</td>
<td>The measure of each angle of an equilateral triangle is ( 60^\circ ).</td>
</tr>
</tbody>
</table>

The converse of the isosceles triangle theorem is the *base angles theorem*.
1. Copy and complete this proof.

**Given** \(FG \parallel HJ; FG \perp FH; JH \perp FH\)

**Prove** \(\angle J \cong \angle G\)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. <em><strong>?</strong></em></td>
<td>1. <em><strong>?</strong></em></td>
</tr>
<tr>
<td>2. (\angle 1 \text{ and } \angle 2 \text{ are right angles.})</td>
<td>2. definition of <em><strong>?</strong></em></td>
</tr>
<tr>
<td>3. (m\angle 1 = 90^\circ; m\angle 2 = 90^\circ)</td>
<td>3. definition of <em><strong>?</strong></em></td>
</tr>
<tr>
<td>4. <em><strong>?</strong></em></td>
<td>4. transitive property of equality</td>
</tr>
<tr>
<td>5. <em><strong>?</strong></em></td>
<td>5. reflexive property</td>
</tr>
<tr>
<td>6. (\triangle JFH \cong \triangle GHF)</td>
<td>6. <em><strong>?</strong></em></td>
</tr>
<tr>
<td>7. <em><strong>?</strong></em></td>
<td>7. <em><strong>?</strong></em></td>
</tr>
</tbody>
</table>

Find the value of \(n\) in each figure.

2. \(\triangle\) with \(76^\circ\), \(8\) cm, \(\angle\), \(8\) cm

3. \(\triangle\) with \(74^\circ\), \(12\) in., \(n\) in., \(\angle\)

4. \(\triangle\) with \(60^\circ\), \(30\) ft, \(n\) ft

**PRACTICE EXERCISES** • For Extra Practice, see page 674.

Find the value of \(x\) in each figure.

5. \(\triangle\) with \(3\) cm, \(2.4\) cm, \(67^\circ\), \(46^\circ\), \(x\) cm

6. \(\triangle\) with \(4\) yd, \(4\) yd, \(4\) yd, \(x\) yd

7. \(\triangle\) with \(10\) m, \(10\) m, \(10\) m, \(x\) m

8. **ARCHITECTURE** An architect sees the figure at the right on a set of building plans. The architect wants to be certain that \(\angle T \cong \angle R\). Copy and complete this proof.

**Given** \(\overline{PS} \cong \overline{QS}; \overline{PT} \cong \overline{QR}\)

Point \(S\) is the midpoint of \(\overline{TR}\).

**Prove** \(\angle T \cong \angle R\)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. <em><strong>?</strong></em></td>
<td>1. <em><strong>?</strong></em></td>
</tr>
<tr>
<td>2. <em><strong>?</strong></em></td>
<td>2. definition of <em><strong>?</strong></em></td>
</tr>
<tr>
<td>3. <em><strong>?</strong></em></td>
<td>3. SSS postulate</td>
</tr>
<tr>
<td>4. (\angle T \cong \angle R)</td>
<td>4. <em><strong>?</strong></em></td>
</tr>
</tbody>
</table>

9. **YOU MAKE THE CALL** A base angle of an isosceles triangle measures \(70^\circ\). Cina says the two remaining angles must each measure \(55^\circ\). What mistake has Cina made?
Name all the pairs of congruent angles in each figure.

10. 11.

12. **BRIDGE BUILDING** On a truss bridge, steel cables cross as shown in the figure below. The inspector needs to be certain that $GL$ and $JK$ are parallel. Copy and complete the proof.

**Given**
- Point $H$ is the midpoint of $GK$.
- Point $H$ is the midpoint of $LJ$.

**Prove** $GL \parallel JK$

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ___ ?__</td>
<td>1. ___ ?__</td>
</tr>
<tr>
<td>2. $GH \cong KH$; ___ ?__</td>
<td>2. ___ ?__</td>
</tr>
<tr>
<td>3. $\angle 1$ and $\angle 2$ are vertical angles.</td>
<td>3. definition of ___ ?__</td>
</tr>
<tr>
<td>4. ___ ?__</td>
<td>4. ___ ?__ theorem</td>
</tr>
<tr>
<td>5. ___ ?__</td>
<td>5. SAS postulate</td>
</tr>
<tr>
<td>6. $\angle G \cong \angle K$, or $m\angle G = m\angle K$</td>
<td>6. ___ ?__</td>
</tr>
<tr>
<td>7. $GL \parallel JK$</td>
<td>7. If ___ ?<strong>, then ___ ?</strong>.</td>
</tr>
</tbody>
</table>

**DATA FILE** For Exercises 13–16, use the data on the types of structural supports used in architecture on page 644. For each type of support, find the measure of each angle in the diagram using a protractor.


**Extended Practice Exercises**

17. Suppose that you join the midpoints of the sides of an isosceles triangle to form a triangle. What type of triangle do you think is formed?

18. **WRITING MATH** Write a proof of the second corollary to the isosceles triangle theorem: The measure of each angle of an equilateral triangle is $60^\circ$.

**Mixed Review Exercises**

Exercises 19–22 refer to the protractor at the right.

(Lesson 3-2)

19. Name the straight angle.
20. Name the three right angles.
21. Name all the obtuse angles and give the measure of each.
22. Name all the acute angles and give the measure of each.
Working with a partner, draw a large acute triangle $ABC$, as shown at the right.

1. With compass point at point $A$, draw two arcs of equal radii that intersect $BC$. Label the points of intersection $X$ and $Y$.

2. Choose a suitable radius of the compass. With compass point first at point $X$, then at point $Y$, draw two arcs that intersect at $Z$.

3. Using a straightedge, draw $AZ$.

4. Label point $D$ where $AZ$ intersects $BC$.

5. Repeat steps 1 through 4, but this time place the compass point at point $B$ and construct a line that intersects $AC$ at point $E$.

6. Repeat steps 1 through 4 again, but now place the compass point at point $C$ and construct a line that intersects $AB$ at point $F$.

7. What observations do you make about the lines you constructed?

**Example 1**

Sketch all the altitudes and medians of $\triangle ABC$.

**Solution**

There are three altitudes, shown below in red.

Similarly, there are three medians, shown below in blue.
Any line, ray, or segment that is perpendicular to a segment at its midpoint is called a **perpendicular bisector** of the segment. In a given plane, however, there is exactly one line perpendicular to a segment at its midpoint. That line usually is called the perpendicular bisector of the segment. The following is an important theorem concerning perpendicular bisectors.

<table>
<thead>
<tr>
<th>The Perpendicular Bisector Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>If a point lies on the perpendicular bisector of a segment, then the point is equidistant from the endpoints of the segment.</td>
</tr>
</tbody>
</table>

You will have a chance to prove this theorem in Exercise 22 on page 167.

**Example 2**

**ARCHITECTURE** A triangular construction is shown on a set of plans. The architect has determined that $DF$ is a perpendicular bisector of $GE$. She needs to know whether the following statements are true or false.

- a. $EF = GF$
- b. $DE = DG$

**Solution**

a. By the definition of perpendicular bisector, $F$ is the midpoint of $GE$. By the definition of midpoint, this means that $EF = GF$, or $EF \equiv GF$. The given statement is true.

b. Point $D$ is a point on the perpendicular bisector of $GE$. By the perpendicular bisector theorem, this means that point $D$ is equidistant from points $G$ and $E$. That is, $DE = DG$, or $DE \equiv DG$. The given statement is true.

Two or more lines that intersect at one point are called **concurrent lines**. You can explore concurrency among the special segments in a triangle by using geometry software or making constructions with a compass and straightedge.

**Example 3**

**GEOMETRY SOFTWARE** Draw a scalene, acute triangle $ABC$. Locate the midpoints of $AB$, $BC$, and $AC$. Draw the three medians of the triangle. What do you notice?

**Solution**

The medians are concurrent. Label the point of concurrence $X$. 
**Try These Exercises**

Trace $\triangle RST$ onto a sheet of paper.

1. Sketch all the altitudes.
2. Sketch all the medians.

In $\triangle XYZ$, $\overline{YW}$ is an altitude. Tell whether each statement is true, false, or cannot be determined.

3. $\overline{YW} \perp \overline{XZ}$
4. $\overline{XW} \equiv \overline{ZW}$
5. $\overline{XY} \equiv \overline{ZY}$
6. $\angle XWY \equiv \angle ZYW$

7. **Geometry Software** Draw a scalene, acute triangle $QPR$. Construct its three altitudes. What do you observe?

8. **Talk About It** Ezra says that an altitude and a median of a triangle could possibly be the same segment. Do you think Ezra’s thinking is correct? Discuss the idea with a partner.

**Practice Exercises** • For Extra Practice, see page 674.

Trace $\triangle JKL$, at the right, onto a sheet of paper.

9. Sketch all the altitudes.
10. Sketch all the medians.

Exercises 11–17 refer to $\triangle EFG$, at the right. Tell whether each statement is true, false, or cannot be determined.

11. $\overline{EG} \equiv \overline{EF}$
12. $\angle EHG \equiv \angle EHF$
13. $\overline{GH} \equiv \overline{FH}$
14. $\angle GEH \equiv \angle FEH$
15. $\overline{EH}$ is median of $\triangle EFG$.
16. $\triangle EGH \equiv \triangle EFH$
17. $\overline{EH}$ is an altitude of $\triangle EFG$.

**Geometric Constructions** Draw two copies of a scalene, acute triangle.

18. Label vertices $A$, $B$, and $C$. Bisect $\angle A$, $\angle B$, and $\angle C$. Label the point of concurrence $Z$. Now measure the perpendicular distance from point $Z$ to each side of the triangle. What do you observe?

19. Draw the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$. Label the point of concurrence $W$. Measure the distance from point $W$ to each vertex of the triangle. What do you observe?

20. **Physics** The center of gravity of an object is the point at which the weight of the object is in perfect balance. Which point of concurrence do you think is the center of gravity of a triangle? Cut a large triangle out of cardboard. Using compass and straightedge, draw medians, altitudes, angle bisectors, and perpendicular bisectors. Place the eraser of a pencil at each point of concurrence. When the triangle balances, you have located the center of gravity.
21. **WRITING MATH** Can a side of a triangle also be an altitude or a median of the triangle? Explain your reasoning.

22. Copy and complete this proof of the perpendicular bisector theorem.

**Given** Line $\ell$ is the perpendicular bisector of $\overline{AC}$. Point $B$ lies on $\ell$.

**Prove** $AB = BC$

<table>
<thead>
<tr>
<th>Statements</th>
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</tr>
</thead>
<tbody>
<tr>
<td>1. ?</td>
<td>1. ?</td>
</tr>
<tr>
<td>2. Point $D$ is the midpoint of $\overline{AC}$.</td>
<td>2. definition of ?</td>
</tr>
<tr>
<td>3. $AD = CD$, or $\overline{AD} \cong \overline{CD}$</td>
<td>3. definition of ?</td>
</tr>
<tr>
<td>4. $\ell \perp \overline{AC}$</td>
<td>4. definition of ?</td>
</tr>
<tr>
<td>5. $\angle 1$ and $\angle 2$ are right angles.</td>
<td>5. definition of ?</td>
</tr>
<tr>
<td>6. $m\angle 1 = 90^\circ$; $m\angle 2 = 90^\circ$</td>
<td>6. definition of ?</td>
</tr>
<tr>
<td>7. $m\angle 1 = m\angle 2$, or $\angle 1 \cong \angle 2$</td>
<td>7. ?</td>
</tr>
<tr>
<td>8. $BD = BD$, or $\overline{BD} \cong \overline{BD}$</td>
<td>8. ?</td>
</tr>
<tr>
<td>10. $\overline{AB} \cong \overline{BC}$, or $AB = BC$</td>
<td>10. ?</td>
</tr>
</tbody>
</table>

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**EXTENDED PRACTICE EXERCISES**

23. **WRITING MATH** Make a list of at least eight true statements concerning $\triangle PQR$, shown at the right.

24. **CHAPTER INVESTIGATION** Imagine your bridge design is to be given to a construction crew. Provide information that will help the crew build the truss accurately. Indicate which line segments are parallel, label angle measures, mark right angles, and classify triangles.

---

**MIXED REVIEW EXERCISES**

Exercises 25–28 refer to the figure below. (Lesson 4-4)

- Name the midpoint of $\overline{AG}$.
- Name the segment whose midpoint is $J$.
- Name all the segments whose midpoint is $E$.
- Assume that $L$ is the midpoint of $BI$. What is its coordinate?

**Given** $f(x) = 3(x - 2)$, **find each value**. (Lesson 3-3)

29. $f(-3)$  
30. $f(2)$  
31. $f(-5)$  
32. $f(8)$  

**Given** $f(x) = -2(x - 3)$, **find each value**. (Lesson 2-2)

33. $f(3)$  
34. $f(-4)$  
35. $f(-2)$  
36. $f(9)$

[mathmatters3.com/self_check_quiz]
Review and Practice Your Skills

Practice Lesson 4-3

Find the value of \( n \) in each figure.

1. \[
\begin{array}{c}
\text{32 cm} \\
\text{8 cm} \\
\text{n cm}
\end{array}
\]

2. \[
\begin{array}{c}
\text{11 ft} \\
\text{60°} \\
\text{60°}
\end{array}
\]

3. \[
\begin{array}{c}
\text{21 in.} \\
\text{n°}
\end{array}
\]

Name all pairs of congruent angles in each figure.

4. \[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}
\]

5. \[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6 \\
7 \\
8 \\
9 \\
10 \\
11 \\
12
\end{array}
\]

Determine whether each statement is **true** or **false**.

6. All equiangular triangles are equilateral.

7. If two sides and one angle in a triangle are congruent to two sides and one angle in another triangle, then the triangles are congruent.

8. The symmetric property applies to both congruent sides and congruent angles.

9. If two triangles share a common side, then they are congruent.

10. Given \( \triangle ABC \cong \triangle DEF \), it can be shown that \( \angle B \cong \angle F \) and \( AB \cong DE \).

Practice Lesson 4-4

Trace each triangle onto a sheet of paper. Sketch all the altitudes and all the medians.

11.

12.

13.

For Exercises 14–19, refer to \( \triangle DEG \) at the right with altitude \( DF \). Tell whether each statement is **true** or **false**.

14. \( \angle E \cong \angle G \)  
15. \( \angle DFE \cong \angle GFD \)  
16. \( m \angle EFD = 90^\circ \)  
17. \( GF \cong EF \)  
18. \( \angle FDG \cong \angle FDE \)  
19. \( DF \perp GE \)

20. Copy and complete this proof.

**Given** \( JK \cong ML; KM \cong LJ \)

**Prove** \( ? \cong \triangle LMJ \)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. _<strong>?</strong></td>
<td>1. Given</td>
</tr>
<tr>
<td>2. ( \overline{JM} \cong \overline{JM} )</td>
<td>2. _<strong>?</strong></td>
</tr>
<tr>
<td>3. _<strong>?</strong> ( \cong \triangle LMJ )</td>
<td>3. SSS Postulate</td>
</tr>
</tbody>
</table>
Find the value of $x$ in each figure. (Lesson 4-1)

21. 

22. 

23. 

Find $m \angle XYZ$ in each figure. (Lesson 4-2)

24. 

25. 

26. 

Mid-Chapter Quiz

Find the unknown measures of the interior angles of each triangle. (Lesson 4-1)

1. an isosceles triangle with an interior angle of 100°
2. a triangle with interior angles of $x^\circ$, $(2x + 10)^\circ$, and $(2x - 5)^\circ$

Sketch each pair of triangles and state either that they are congruent or that no conclusion is possible. If they are congruent, name the postulate that could be used to prove the congruence. (Lesson 4-2)

3. Triangles $ABD$ and $CBD$ share side $BD$. Side $BD$ is the perpendicular bisector of $AC$.
4. Triangles $EFH$ and $GFH$ share side $FH$. Sides $EF$ and $GF$ are congruent. Angles $E$ and $G$ are congruent.
5. Triangles $RSU$ and $TSU$ share side $SU$. Side $SU$ bisects angles $RST$ and $RUT$.

Determine whether each statement is true or false. (Lesson 4-3)

6. Two isosceles triangles with congruent vertex angles always have congruent base angles.
7. If two sides of a triangle are congruent, then the base angles and the vertex angle must be congruent.

Determine whether each statement is always, sometimes, or never true. (Lesson 4-4)

8. An altitude of a triangle divides the corresponding side of the triangle into two congruent parts.
9. The median of a side of a triangle is perpendicular to that side of the triangle.
10. An altitude of a triangle is a segment that is inside the triangle.
The proofs that you have studied so far in this book have been direct proofs. That is, the proofs proceeded logically from a hypothesis and known facts to show that a desired conclusion is true. In this lesson, you will study indirect proof. In an indirect proof, you begin with the desired conclusion and assume that it is not true. You then reason logically until you reach a contradiction of the hypothesis or of a known fact.

**Problem**

**Problem Statement**

Prove the following statement.

*If a figure is a triangle, then it cannot have two right angles.*

**Solve the Problem**

Begin by drawing a representative triangle, such as \( \triangle ABC \) at right.

**Step 1**

Assume that the conclusion is false. That is, assume that a triangle can have two right angles. In particular, in \( \triangle ABC \), assume that \( \angle A \) and \( \angle B \) are right angles.

**Step 2**

Reason logically from the assumption, as follows. By the definition of a right angle, \( m\angle A = 90^\circ \) and \( m\angle B = 90^\circ \). By the addition property of equality, \( m\angle A + m\angle B = 90^\circ + 90^\circ = 180^\circ \). By the protractor postulate, \( m\angle C = n^\circ \), where \( n \) is a positive number less than or equal to 180. By the addition property of equality, \( m\angle A + m\angle B + m\angle C = 180^\circ + n^\circ \). By the triangle-sum theorem, \( m\angle A + m\angle B + m\angle C = 180^\circ \).

**Step 3**

Note that the last two statements in Step 2 are contradictory. Therefore, the assumption that a triangle can have two right angles is false. The given statement must be true.

The solution of the problem above illustrates the following general method for writing an indirect proof.

**Writing an Indirect Proof**

- **Step 1** Assume temporarily that the conclusion is false.
- **Step 2** Reason logically until you arrive at a contradiction of the hypothesis or a contradiction of a known fact (a definition, a postulate, or a previously proved theorem).
- **Step 3** State that the temporary assumption must be false, and that the given statement must be true.
TRY THESE EXERCISES

Suppose you are asked to write an indirect proof of each statement. Write Step 1 of the proof.

1. **ART** If the triangle in a sculpture is a right triangle, then it cannot be an obtuse triangle.

2. **ARCHITECTURE** If the triangle in a building design is equilateral, then it is an isosceles triangle.

PRACTICE EXERCISE

Copy and complete the indirect proof of each theorem.

3. Theorem: If two lines intersect, then they intersect in one point.
   
   **Step 1:** Assume that two lines can intersect in ___?__ points. In particular, in the figure at the right, assume that there are lines ____?__ and ____?__ that intersect at points ____?__ and ____?__.
   
   **Step 2:** By the unique line postulate (postulate 1), there is exactly ___?__ line through points X and Y.
   
   **Step 3:** The statements in Step 1 and Step 2 are ___?__. Therefore, the assumption that two lines can intersect in two points is ___?__. The given statement must be ___?__.

4. Theorem: Through a point not on a line, there is exactly one line parallel to the given line.
   
   **Step 1:** Assume that there are ___?__ lines parallel to the given line. In particular, in the figure at the right, assume that, through point P, ____?__ || ℓ and ____?__ || ℓ.
   
   **Step 2:** By the parallel lines postulate, m∠ ____?__ = m∠3 and m∠ ____?__ = m∠3. By the transitive property of equality, m∠ ____?__ = m∠ ____?__.
   
   However, because m and n are different lines, m∠1 ≠ m∠2.
   
   **Step 3:** The last two statements in Step 2 are ___?__. Therefore, the assumption that there can be two lines parallel to a given line through a point outside the line is ___?__. The given statement must be ___?__.

5. Write an indirect proof of the following theorem. Through a point outside a line, there is exactly one line perpendicular to the given line. (Hint: Use the proof in Exercise 4 above as a model.)

6. **WRITING MATH** Write what you would do to prove indirectly that a triangle cannot have two obtuse angles.

MIXED REVIEW EXERCISES

Simplify each expression. (Lesson 1-4)

7. \(-3 + 4 – (-6) + (-2)\)  
8. \(-9 + (-4) – (-3) + 8\)  
9. \(4 – (-6) + 2 + (-3)\)

10. \((4) + 9 – 3 – (-2)\)  
11. \(8 – (-3) + 2 – 3 + 9\)  
12. \(2 – (-8) – (-(-12))\)
Inequalities in Triangles

Goals
- Understand relationships among sides and angles of a triangle.

Applications
- Construction, Art, Architecture

Work in groups of two or three students.

The figure at the right shows four paths that ants took from point A to point B.

1. Using a centimeter ruler, find the length of each path. (You will need to use some ingenuity to measure path 2!)
2. Trace points A and B onto a sheet of paper. Can you draw a path from point A to point B that is longer than any of the given paths? Use the ruler to find the length of your path.
3. Can you draw a path from point A to point B that is shorter than any of the given paths? Use the ruler to find the length of your path.

Build Understanding

In the activity above, you had an opportunity to investigate yet another fundamental postulate of geometry.

Postulate 14

The Shortest Path Postulate

The length of the segment that connects two points is shorter than the length of any other path that connects the points.

The shortest path postulate leads to some important conclusions about triangles. As an example, consider the following proof.

Given \( \triangle ABC \)

Prove \( AB + BC > AC \)

Proof

Assume that \( AB + BC \nsim AC \). Then one of these two statements must be true:

\( AB + BC = AC \) or \( AB + BC < AC \).

If \( AB + BC = AC \), then there is a path other than AC that connects points A and C that is equal to AC; this contradicts the shortest path postulate.

Similarly, if \( AB + BC < AC \), then there is a path connecting points A and C that is shorter than AC; this also contradicts the shortest path postulate.

Therefore, the assumption \( AB + BC \nsim AC \) must be false. It follows that the desired conclusion, \( AB + BC > AC \), is true.
In Exercises 23 and 24 on page 174, you will prove that $AB + AC > BC$ and $AC + BC > AB$ are true statements also. So, you will have completed the proof of the following theorem.

| The Triangle Inequality Theorem | The sum of the lengths of any two sides of a triangle is greater than the length of the third side. |

**Example 1**

**CONSTRUCTION** A frame must be built to pour a triangular cement slab to complete a walkway. The lengths of two sides of the triangle are 5 ft and 9 ft. Find the range of possible lengths for the third side.

**Solution**

Use the variable $n$ to represent the length in feet of the third side. By the triangle inequality theorem, these three inequalities must be true.

I. $5 + 9 > n$  
   II. $5 + n > 9$  
   III. $9 + n > 5$

14 > $n$  
$n$ > 4  
$n$ > -4

Inequality III is not useful, since a length must be a positive number. From inequalities I and II, you obtain the combined inequality 14 > $n$ > 4.

So, the length of the third side must be less than 14 ft and greater than 4 ft.

The following two theorems also involve inequalities in triangles. In this book, we will accept these theorems as true without proof.

| The Unequal Sides Theorem | If two sides of a triangle are unequal in length, then the angles opposite those sides are unequal in measure, in the same order. |
| The Unequal Angles Theorem | If two angles of a triangle are unequal in measure, then the sides opposite those angles are unequal in length, in the same order. |

**Example 2**

In $\triangle KLM$, $KL = 8$ in., $LM = 10$ in., and $KM = 7$ in. List the angles of the triangle in order from largest to smallest.

**Solution**

Draw and label $\triangle KLM$, as shown at the right.

The angle opposite $LM$ is $\angle K$.  
The angle opposite $KL$ is $\angle M$.

Since 10 > 8, $LM > KL$.  
So, by the unequal sides theorem, $m\angle K > m\angle M$.  
By similar logic, $m\angle M > m\angle L$.  
So, from largest to smallest, the angles are $\angle K$, $\angle M$, and $\angle L$. 
TRY THESE EXERCISES

ART The design for a sculpture has three triangular platforms. The lengths of two sides of each platform are given. Find the range of possible lengths for the third side.

1. 6 ft, 9 ft
2. 7 ft, 7 ft
3. 2 ft, 7 ft

List the angles of each triangle in order from largest to smallest.

4. 5. 6.

In $\triangle XYZ$, $\angle X = 56^\circ$ and $\angle Z = 19^\circ$. List the sides of the triangle in order from shortest to longest.

8. ARCHITECTURE The base for an indoor fountain has a triangular shape. On the plans, the base is shown as $\triangle RST$. If $\angle S > \angle R$ and $\angle R > \angle T$, which is the shortest side of the triangle?

PRACTICE EXERCISES • For Extra Practice, see page 675.

Determine if the given measures can be lengths of the sides of a triangle?

9. 7 cm, 2 cm, 6 cm
10. 7.3 m, 15 m, 7.3 m
11. $9\frac{1}{4}$ ft, $3\frac{1}{2}$ ft, $5\frac{3}{4}$ ft
12. 24 in., 5 ft, 54 in.
13. 34 yd, 34 yd, 34 yd
14. 3 mm, 5 cm, 7 mm

Which is the longest side of each triangle? the shortest?

15. $\triangle DEF$
16. $\triangle QPR$
17. $\triangle UVW$

In each figure, give a range of possible values for $x$.

18. $x \text{ m}$
19. $x \text{ ft}$
20.

21. In $\triangle CDE$, $CD < DE$ and $CE < CD$. Which is the largest angle of the triangle?

22. GEOMETRY SOFTWARE Use the following information to draw $\triangle QRS$: $QS = 17$, $RS = 23$, and $QR = 20.5$. List the angles of the triangle in order from largest to smallest.

For Exercises 23 and 24, refer to the proof on page 172.

23. Given $\triangle ABC$  24. Given $\triangle ABC$
Prove $AB + AC > BC$  Prove $AC + BC > AB$
List all the segments in each figure in order from longest to shortest.

25. [Diagram]

Give a range of possible values for \( z \).

27. [Diagram]

28. [Diagram]

EXTENDED PRACTICE EXERCISES

29. **WRITING MATH** In a right triangle, the side opposite the right angle is called the *hypotenuse*. Explain why the hypotenuse must be the longest side.

30. **ERROR ALERT** A blueprint calls for the construction of a right triangle with sides measuring 5 ft, 6 ft, and 11 ft. How do you know the measurements are incorrect?

**CONSTRUCTION** Manuella is building an A-frame dog house with the front in the shape of an isosceles triangle. Two sides of the front will each be 4 ft long.

31. Under what conditions will the base of the front of the dog house be exactly 4 ft?

32. Under what conditions will the base of the front of the dog house be greater than 4 ft?

33. Under what conditions will the base of the front of the dog house be shorter than 4 ft?

34. **CHAPTER INVESTIGATION** Using your design for a truss bridge, build a section of the truss using straws or toothpicks. Use a ruler and protractor to make sure your construction matches the plans.

MIXED REVIEW EXERCISES

Write a function rule to represent the number of points in the \( n \)th figure in the patterns below. (Lesson 3-5)

35. [Pattern]

36. [Pattern]

Write each number in scientific notation. (Lesson 1-8)

37. 371,000,000,000

38. 0.000000074

39. 256,000,000,000

40. 0.000000942

41. 8,900,000,000,000

42. 0.000000007
Write Step 1 of an indirect proof of each statement.
1. If a triangle is not isosceles, then it is not equilateral.
2. If a point lies on the perpendicular bisector of a segment, then the point is equidistant from the endpoints of the segment.
3. If two angles are vertical angles, then they are equal in measure.
4. If two parallel lines are cut by a transversal, then alternate interior angles are equal in measure.
5. If two lines are perpendicular, then they intersect.
6. The sum of the measures of the angles of a triangle is 180°.

Write an indirect proof of each statement.
7. If a triangle is a right triangle, then it cannot be an obtuse triangle.
8. If a triangle is equilateral, then it is isosceles.
9. If two angles are vertical angles, then they are equal in measure.
10. If two sides of a triangle are not congruent, then the angles opposite those sides are not congruent.

Can the given measures be the lengths of the sides of a triangle?
11. 5.5 ft, 8.2 ft, 12.9 ft
12. 14 cm, 35 cm, 21 cm
13. 21 m, 13.2 m, 7 m

In each figure, give a range of possible values for \( x \).
14. \( 12.5 \text{ m} \)
15. \( 7 \text{ ft} \)
16. \( 3.4 \text{ yd} \)

List all the segments in each figure in order from shortest to longest.
17. \( P, Q, R, M, N, O, M, N, O \)

Determine whether each statement is true or false.
20. In a scalene triangle, no two angles are equal in measure. (Lesson 4-6)
21. A triangle can have sides of length 178 cm, 259 cm, and 440 cm. (Lesson 4-6)
Determine whether each statement is true or false.

22. All equilateral triangles are also isosceles triangles. (Lesson 4-1)
23. If \( \triangle ABC \cong \triangle DEF \), then \( \triangle BAC \cong \triangle EDF \). (Lesson 4-2)
24. If two angles of a triangle are congruent, then the triangle is isosceles.
25. All altitudes of a triangle lie in the interior of the triangle. (Lesson 4-4)
26. In an indirect proof, one starts by assuming that the conclusion is false. (Lesson 4-5)

Find the value of \( x \) in each figure. (Find the range of possible values for \( x \) in Exercise 32.)

27. (Lesson 4-1) 

28. (Lesson 4-1) 

29. (Lesson 4-2) 

30. (Lesson 4-3) 

31. (Lesson 4-4) 

32. (Lesson 4-6) 

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MathWorks Career – Animator

Traditional animation involves making many hand-drawn pictures with slight differences and filming them frame by frame to create the illusion of motion. The newest form of animation is computer-assisted animation. Knowledge of coordinates, area of curved surfaces, conics and polygons are all important pieces of an animator’s tool kit for drawing great pictures.

To give objects depth, animators use perspective drawing. For instance, to make a house look three-dimensional, it must be drawn so that the house’s front walls look larger than those in the rear of the house.

1. The front wall of the house in the drawing has a perimeter of \( 6\frac{1}{4} \) in. Find the measure of \( x \).
2. The roof panels and side wall shown are drawn as parallelograms. Find the measures of \( a, b, c, \) and \( d \).
3. The altitude of the triangle formed by the roof is 0.5 in. Find the length of the sides of the triangle to the nearest hundredth inch.

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Work with a partner.

Draw and label a pentagon as shown at the left below. Then cut out the five exterior angles and arrange them as shown at the right.

1. What is the relationship among the five exterior angles?

2. Repeat the experiment, this time drawing a hexagon and labeling six exterior angles. What is the relationship among the exterior angles?

A polygon is a closed plane figure that is formed by joining three or more coplanar segments at their endpoints. Each segment is called a side of the polygon. Each side intersects exactly two other sides, one at each endpoint. The point at which two sides meet is called a vertex of the polygon.

A polygon is convex if each line containing a side contains no points in the interior of the polygon. A polygon that is not convex is called concave. In this book, when the word polygon is used, assume the polygon is convex. The angles determined by the sides are called the angles, or the interior angles, of the polygon.

Two sides of a polygon that intersect are called consecutive sides. The endpoints of any side of a polygon are consecutive vertices. When naming a polygon, you list consecutive vertices in order. For example, two names for the pentagon at the right are “pentagon ABCDE” and “pentagon BCDEA.” It is not correct to call the figure “pentagon ABCED.”

A diagonal of a polygon is a segment that joins two nonconsecutive vertices. In pentagon ABCDE, the diagonals are shown in red.
If you draw all the diagonals from just one vertex of a polygon, you divide the interior of the polygon into nonoverlapping triangular regions. The sum of the measures of the angles of the polygon is the product of the number of triangular regions formed and 180°.

In each case, the number of triangular regions formed is two fewer than the number of sides of the polygon. This leads to the following theorem.

**Example 1**

**SURVEYING** A playground has the shape shown in the figure to the right. A surveyor measures six of the angles of the playground. Find the unknown measure.

**Solution**

The polygon has 7 sides. Use the polygon-sum theorem to find the sum of the angle measures.

\[(n – 2)180° = (7 – 2)180° = (5)180° = 900°\]

Add the known angle measures.

\[139° + 124° + 144° + 130° + 118° + 125° = 780°\]

Subtract this sum from 900°: 900° – 780° = 120°

The unknown angle measure is 120°.

An **exterior angle** of any polygon is an angle both adjacent to and supplementary to an interior angle. Since the sum of the interior angles of a polygon depends on the number of sides of the polygon, you might expect that the same would be true for the exterior angles. So, the following theorem about exterior angles may come as a surprise to you.

**The Polygon Exterior Angle Theorem**

The sum of the measures of the exterior angles of a convex polygon, one angle at each vertex, is 360°.

A polygon with all sides of equal length is called an **equilateral polygon**. A polygon with all angles of equal measure is an **equiangular polygon**. A **regular polygon** is a polygon that is *both* equilateral and equiangular.
Example 2

a. Find the measure of each interior angle of a regular octagon.
b. Find the measure of each exterior angle of a regular octagon.

Solution

a. Using the polygon-sum theorem, the sum of the measures of the interior angles is \((n - 2)180^\circ = (8 - 2)180^\circ = 6 \times 180^\circ = 1080^\circ\).

Because the octagon is regular, the interior angles are equal in measure.

So, the measure of one interior angle is \(1080^\circ \div 8 = 135^\circ\).

b. By the polygon exterior angle theorem, the sum of the measures of the exterior angles is \(360^\circ\).

So, the measure of one exterior angle is \(360^\circ \div 8 = 45^\circ\).

Try These Exercises

Find the unknown angle measure or measures in each figure.

1. 2. 3.

4. Find the measure of each interior angle of a regular polygon with 15 sides.
5. Find the measure of each exterior angle of a regular decagon.

Practice Exercises • For Extra Practice, see page 675.

Find the unknown angle measure or measures in each figure.

6. 7. 8.

9. 10. 11.

12. A road sign is in the shape of a regular hexagon. Find the measure of each interior angle.
13. **RECREATION** A game board is in the shape of a regular polygon with 18 sides. Find the sum of the measures of the interior angles.

14. Find the sum of the measures of the exterior angles of a regular nonagon.

15. Find the measure of each exterior angle of a regular polygon with 24 sides.

Each figure is a regular polygon. Find the values of $x$, $y$, and $z$.

16. 

17. 

18. 

Find the number of sides of each regular polygon.

19. The measure of each exterior angle is $9^\circ$.

20. The sum of the measures of the interior angles is $1980^\circ$.

21. The measure of each interior angle is $162^\circ$.

For Exercises 22–23, use the Reading Math feature on page 221 to locate information about convex regular polyhedrons.

22. A **polyhedron** is a closed three-dimensional figure in which each surface is a polygon. Why do you think these are called **regular** polyhedrons?

23. **SPORTS** At the right is a soccer ball. It is shaped like a polyhedron with faces that are all regular polygons. However, this shape is not pictured with the convex regular polyhedrons on page 221. Explain.

**EXTENDED PRACTICE EXERCISES**

For Exercises 24 and 25, consider a regular polygon with $n$ sides. Write an expression to represent each quantity.

24. the measure in degrees of one exterior angle

25. the measure in degrees of one interior angle

**WRITING MATH** For Exercises 26–28, consider what happens as the number of sides of a regular polygon becomes larger and larger.

26. What happens to the measure of each exterior angle?

27. What happens to the measure of each interior angle?

28. What happens to the overall appearance of the polygon?

**MIXED REVIEW EXERCISES**

Classify each triangle by its sides and angles.

29. 

30. 

31. 

32. 

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Special Quadrilaterals: Parallelograms

Goals  ■ Apply properties of parallelograms to find missing lengths and angle measures.

Applications  Art, Construction, Engineering, Architecture

On a sheet of paper, draw line $\ell$. Mark and label a point $A$ on line $\ell$ as shown at the right.

1. With compass point at point $A$, draw two arcs of equal radii that intersect $\ell$. Label the points of intersection $X$ and $Y$.

2. With compass point first at point $X$, then at point $Y$, draw two arcs that intersect at $Z$.

3. Using a straightedge, draw $\overline{AZ}$. What do you observe about the line you constructed?

4. Use this method to construct a rectangle. Using a straightedge, draw the diagonals of your rectangle. What observations do you make about the diagonals?

**BUILD UNDERSTANDING**

Opposite sides of a quadrilateral are two sides that do not share a common endpoint. Opposite angles are two angles that do not share a common side.

A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

The following theorems identify some properties of all parallelograms.

<table>
<thead>
<tr>
<th>The Parallelogram-Side Theorem</th>
<th>If a quadrilateral is a parallelogram, then its opposite sides are equal in length.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The Parallelogram-Angle Theorem</td>
<td>If a quadrilateral is a parallelogram, then its opposite angles are equal in measure.</td>
</tr>
<tr>
<td>The Parallelogram-Diagonal Theorem</td>
<td>If a quadrilateral is a parallelogram, then its diagonals bisect each other.</td>
</tr>
</tbody>
</table>
The proofs of these theorems are based on properties of parallel lines and congruent triangles. You will have a chance to prove them in Exercises 22–25 on page 185.

**Example 1**

Find \( m \angle J \) in \( \square JKLM \).

**Solution**

Since \( \angle K \) and \( \angle M \) are opposite angles, by the parallelogram-angle theorem, \( m \angle K = m \angle M = 48^\circ \).

Use the polygon-sum theorem to find the sum of the measures of the interior angles.

\[
(n - 2)180^\circ = (4 - 2)180^\circ = 2(180^\circ) = 360^\circ
\]

Notice that \( m \angle M + m \angle K = 48^\circ + 48^\circ = 96^\circ \). It follows that \( m \angle J + m \angle L = 360^\circ - 96^\circ = 264^\circ \).

Since \( \angle J \) and \( \angle L \) are opposite angles, by the parallelogram-angle theorem, \( m \angle J = 264^\circ / 2 = 132^\circ \).

Other special quadrilaterals are rectangles, rhombuses, and squares.

A **rectangle** is a quadrilateral with four right angles.

A **rhombus** is a quadrilateral with four sides of equal length.

A **square** is a quadrilateral with four right angles and four sides of equal length.

Rectangles, rhombuses, and squares are parallelograms, and so they have all the properties of parallelograms. In addition, however, they have the special properties summarized in the following theorems. In this book, these theorems will be accepted as true without proof.

**The Rectangle-Diagonal Theorem**

If a quadrilateral is a rectangle, then its diagonals are equal in length.

**The Rhombus-Diagonal Theorem**

If a quadrilateral is a rhombus, then its diagonals are perpendicular and bisect each pair of opposite angles.

**Example**

A rectangular mural is reinforced from the back using wire diagonals. The diagram at the right shows how the wires are attached. If \( ZO = 8 \text{ ft} \), find \( WY \).
Solution

A rectangle is a parallelogram. By the parallelogram-diagonal theorem, the diagonals bisect each other. So, \( XZ = 2(ZO) = 2(8 \text{ ft}) = 16 \text{ ft} \).

Then, by the rectangle-diagonal theorem, you know that the diagonals are equal in length. So, \( WY = XZ = 16 \text{ ft} \).

Try These Exercises

In Exercises 1–2, each figure is a parallelogram. Find the values of \( x \) and \( z \).

1.  

   \[
   \begin{align*}
   x^\circ & \quad z^\circ \\
   112^\circ & \\
   
   \end{align*}
   \]

2.  

   \[
   \begin{align*}
   1.4 \text{ cm} & \quad x \text{ cm} \\
   2.5 \text{ cm} & \quad z \text{ cm} \\
   
   \end{align*}
   \]

BRIDGE BUILDING  A portion of a truss bridge forms quadrilateral \( XYZW \), shown at the right. Given that \( XYZW \) is a rhombus and \( m\angle YXZ = 32^\circ \), find the measure of each angle.

3. \( \angle YXW \)

4. \( \angle XYW \)

5. \( \angle XWY \)

6. \( \angle YZW \)

7. \( \angle YVZ \)

8. \( \angle XWZ \)

Practice Exercises  • For Extra Practice, see page 676.

ARCHITECTURE  The parallelograms in Exercises 9–12 are from building plans. Find the values of \( a, b, c, \) and \( d \).

9.  

   \[
   \begin{align*}
   \text{28 in.} & \quad 135^\circ \\
   \text{42 in.} & \quad b^\circ \\
   \text{a in.} & \quad c \text{ in.} \\
   
   \end{align*}
   \]

10.  

    \[
    \begin{align*}
    8 \text{ cm} & \quad a \text{ cm} \\
    8 \text{ cm} & \quad 70^\circ \\
    \end{align*}
    \]

11. \( MJ = 1\frac{1}{2} \text{ yd}, MK = 3\frac{1}{4} \text{ yd} \)

12. \( RS = 4.9 \text{ mm}, RQ = 5.6 \text{ mm}, RP = 9.7 \text{ mm} \)

ERROR ALERT  Dillon made the following statements about quadrilaterals. Decide whether each statement is true or false.

13. A rectangle is a parallelogram.

14. No rhombus is a square.

15. Every quadrilateral is a parallelogram.

16. Some rectangles are rhombuses.

17. The diagonals of a square are not equal in length.

18. Consecutive angles of a parallelogram are supplementary.
WRITING MATH  Do you think that the given figure is a parallelogram? Write yes or no. Then explain your reasoning.

19. 20. 21. 

22. Copy and complete this proof.

Given   \(ABCD\) is a parallelogram.

Prove   \(m\angle A = m\angle C\)

<table>
<thead>
<tr>
<th>Statements</th>
<th>Reasons</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ___ ?_</td>
<td>1. ___ ?_</td>
</tr>
<tr>
<td>2. (AB \parallel DC; AD \parallel BC)</td>
<td>2. definition of _<strong>?</strong></td>
</tr>
<tr>
<td>3. (m\angle 1 = m\angle 3), or (\angle 1 \cong \angle 3) (m\angle 2 = m\angle 4), or (\angle 2 \cong \angle 4)</td>
<td>3. If ___ ?__, then _<strong>?</strong></td>
</tr>
<tr>
<td>4. ___ ?_</td>
<td>4. reflexive property</td>
</tr>
<tr>
<td>5. ___ ?_</td>
<td>5. ASA postulate</td>
</tr>
<tr>
<td>6. (m\angle A = m\angle C)</td>
<td>6. ___ ?_</td>
</tr>
</tbody>
</table>

23. The proof in Exercise 22 is the beginning of a proof of the parallelogram-angle theorem. Using this proof as a model, write the second part of the proof. That is, prove \(m\angle B = m\angle D\).

24. Write a proof of the parallelogram-side theorem.

**Extended Practice Exercises**

25. WRITING MATH  Suppose that you are asked to prove the parallelogram-diagonal theorem. Write a paragraph that explains how you would proceed. (Do not write the two-column proof.)

26. DESIGN  Suppose you need to describe the figure at the right to a graphics designer. State as many facts as you can about the figure.

**Mixed Review Exercises**

Refer to the figure at the right for Exercises 27–29.

27. Name all the alternate exterior angles.

28. Name all the corresponding angles.

29. Name all the alternate interior angles.

Determine if each relation is a function. Give the domain and range.

30. \(a\) 0 1 2 2 3 31. \(x\) 2 3 4 5 6 32. \(m\) -1 0 1 0 -1
\(b\) -1 -3 -4 -5 -6 \(y\) 4.5 6.5 8.5 10.5 12.5 \(n\) -4 -1 0 2 5

mathmatters3.com/self_check_quiz
Find the unknown angle measure or measures in each figure.

1. \[ \begin{align*} \text{m}^\circ & = 59^\circ \\ 68^\circ & \end{align*} \]

2. \[ \begin{align*} \text{n}^\circ & = 72^\circ \\ 108^\circ & \end{align*} \]

3. \[ \begin{align*} x^\circ & \end{align*} \]

4. Find the measure of each interior angle of a regular polygon with 13 sides.
5. Find the measure of each exterior angle of a regular polygon with 20 sides.
6. Find the sum of the measures of the interior angles of a regular heptagon.
7. Find the sum of the measures of the exterior angles of a regular heptagon.
8. Using diagonals from one vertex, into how many nonoverlapping triangular regions can you divide a nonagon? a polygon with 21 sides?

Find the number of sides of each regular polygon.

9. The measure of each exterior angle is 40°.
10. The sum of the measures of interior angles is 2160°.
11. The measure of each interior angle is 165°.

Determine whether each statement is true or false.

12. The diagonals of a rhombus are equal in length.
13. Every square is a rhombus.
14. Quadrilaterals include squares, parallelograms, pentagons, and rectangles.
15. A square is a regular polygon.
16. In all quadrilaterals, the opposite sides are equal in length.

For the following parallelograms, find the values of \( a, b, c, \) and \( d. \)

17. \[ \begin{align*} \text{c cm} & = 43 \text{ cm} \\ a^\circ & = 115^\circ \\ d \text{ cm} & = 37 \text{ cm} \end{align*} \]

18. \[ \begin{align*} \text{c ft} & = 65 \text{ ft} \\ a^\circ & = 18 \text{ ft} \\ b^\circ & = 18 \text{ ft} \end{align*} \]

19. \[ \begin{align*} d^\circ & = 133^\circ \\ a \text{ m} & = 50 \text{ m} \\ b \text{ m} & = 6.8 \text{ m} \\ c \text{ m} & = 8.8 \text{ m} \end{align*} \]

Is the given figure a parallelogram? Write yes or no. Then explain your reasoning.

20. \[ \begin{align*} 8.3 & \\ 27 & \\ 8.5 & \end{align*} \]

21. \[ \begin{align*} 5 & \\ 3 & \\ 5 & \end{align*} \]

22. \[ \begin{align*} 8.3 & \\ 27 & \\ 8.5 & \end{align*} \]
Find the value of $x$ in each figure. (Lesson 4-1)

23. 

24. 

25. 

26. Copy and complete this proof. (Lesson 4-2)

Given $AE \cong EC; DE \cong EB$

Prove $\triangle DAE \cong \triangle$ ?

Statements Reasons
1. ___?__ 1. Given
2. ___?__ 2. Vertical Angles Theorem
3. $\triangle DAE \cong$ ? 3. ___ ?__

Find the value of $n$ in each figure. (Lesson 4-3)

27. 

28. 

29. 

Give the range of possible values for $x$ in each figure. (Lesson 4-6)

30. 

31. 

32. 

Find the unknown angle measure or measures in each figure. (Lesson 4-7)

33. 

34. 

35. 

36. 

37. 

38.
Work with a partner.

The *tangram* is an ancient Chinese puzzle consisting of the seven pieces shown at the right. Use a manufactured set of tangram pieces or trace the figure onto a sheet of paper and then cut out the pieces along the lines.

1. Arrange pieces *E*, *F*, and *G* to form a rectangle.
2. Arrange *E*, *F*, and *G* to form a parallelogram that is not a rectangle.
3. Arrange *E*, *F*, and *G* to form a quadrilateral that is not a parallelogram.
4. Arrange pieces *A*, *C*, *E*, and *G* to form a square.
5. Arrange all seven tangram pieces to form a quadrilateral that is not a parallelogram.
6. Form as many different rectangles that are not squares as possible. (For each rectangle, use as many tangram pieces as needed.)

**BUILD UNDERSTANDING**

A *trapezoid* is a quadrilateral with exactly one pair of parallel sides. The parallel sides are called the **bases** of the trapezoid. Two consecutive angles that share a base form a pair of **base angles**; every trapezoid has two pairs of base angles. The nonparallel sides are called the **legs**.

The **median** of a trapezoid is the segment that joins the midpoints of the legs. Two important properties of the median are stated in the following theorem, which will be accepted as true without proof.

<table>
<thead>
<tr>
<th>The Trapezoid-Median Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>If a segment is the median of a trapezoid, then it is:</td>
</tr>
<tr>
<td>1. parallel to the bases; and</td>
</tr>
<tr>
<td>2. equal in length to one half the sum of the lengths of the bases.</td>
</tr>
</tbody>
</table>
Example 1

STAGE DESIGN  The plans for two panels of a stage setting are shown in the figure at the right. In the figure, $QT \parallel RS$. Find $AB$.

Solution

Quadrilateral $QRST$ is a trapezoid. $QT$ and $RS$ are the bases, and $AB$ is the median. To find $AB$, apply the trapezoid-median theorem.

\[
AB = \frac{1}{2}(QT + RS)
\]

\[
AB = \frac{1}{2}(16 + 25)
\]

\[
AB = \frac{1}{2}(41)
\]

So, the length of $AB$ is 20.5 cm.

A trapezoid with legs of equal length is called an isosceles trapezoid.

The following theorem states an important fact about isosceles trapezoids. This theorem also will be accepted as true without proof.

The Isosceles Trapezoid Theorem

If a quadrilateral is an isosceles trapezoid, then its base angles are equal in measure.

Example 2

In the figure at the right, $ST \parallel VW$. Find $m\angle V$.

Solution

Quadrilateral $STVW$ is an isosceles trapezoid, with bases $ST$ and $VW$. So, $\angle W$ and $\angle V$ are a pair of base angles, and they are equal in measure. Use this fact to write and solve an equation.

\[
a + 27 = 3a - 57
\]

Add $-a$ to each side.

\[
a + 27 + (-a) = 3a - 57 + (-a)
\]

Combine like terms.

\[
27 = 2a - 57
\]

Add 57 to each side.

\[
27 + 57 = 2a - 57 + 57
\]

\[
84 = 2a
\]

Multiply each side by $\frac{1}{2}$.

\[
42 = a
\]

So, the value of $a$ is 42. From the figure, $m\angle V = (3a - 57)^\circ$.

Substituting 42 for $a$, $m\angle V = (3 \cdot 42 - 57)^\circ = (126 - 57)^\circ = 69^\circ$. 

Check Understanding

In Example 2, what is the measure of $\angle S\angle T$?
**TRY THESE EXERCISES**

A trapezoid and its median are shown. Find the value of $x$.

1. 14 ft
   \[ \frac{x}{18} \]  

   2. \[ 9 \text{ cm} \]
   \[ \frac{6.5 \text{ cm}}{x} \]

   3. \[ 7 \text{ m} \]
   \[ \frac{(x + 2) \text{ m}}{3} \]

**CONSTRUCTION** The given figures are part of a design for a wrought-iron railing. Find all unknown angle measures.

4. \[ P \]
   \[ R \]
   \[ 126^\circ \]

   5. \[ C \]
   \[ D \]
   \[ (4n + 27)^\circ \]
   \[ (3n + 39)^\circ \]

**PRACTICE EXERCISES** • For Extra Practice, see page 676.

A trapezoid and its median are shown. Find the value of $z$.

6. \[ 38 \text{ in.} \]
   \[ \frac{z}{27} \text{ in.} \]

   7. \[ 4.9 \text{ m} \]
   \[ \frac{z}{2.3} \text{ m} \]

   8. \[ 2\frac{3}{4} \text{ ft} \]
   \[ \frac{4 \text{ ft}}{z} \text{ ft} \]

   9. \[ 14 \text{ yd} \]
   \[ \frac{z}{2.5} \text{ yd} \]

   10. \[ 17 \text{ in.} \]
   \[ \frac{(z - 4) \text{ in.}}{25} \text{ in.} \]

   11. \[ z \text{ mm} \]
   \[ \frac{14 \text{ mm}}{3z \text{ mm}} \]

The given figure is a trapezoid. Find all the unknown angle measures.

12. \[ G \]
   \[ H \]
   \[ 61^\circ \]

   13. \[ T \]
   \[ U \]
   \[ (6a - 31)^\circ \]
   \[ (4a + 7)^\circ \]

14. **CHAPTER INVESTIGATION** Make a list of the quadrilaterals that you can see in your truss bridge design. Compare your design with those of your classmates. Which design do you think will support the most weight? Why?

15. **WRITING MATH** Compare the median of a trapezoid to the median of a triangle. How are they alike? How are they different?
In Exercises 16–21, give as many names as are appropriate for the given figure. Choose from quadrilateral, parallelogram, rhombus, rectangle, square, trapezoid, and isosceles trapezoid. Then underline the best name for the figure.

16.

17.

18.

19.

20.

21.

Copy and complete the following table that summarizes what you have learned about quadrilaterals. For each entry, write yes or no.

<table>
<thead>
<tr>
<th>Property</th>
<th>Quadrilateral</th>
<th>Parallelogram</th>
<th>Rectangle</th>
<th>Rhombus</th>
<th>Square</th>
<th>Trapezoid</th>
<th>Isos. Trap.</th>
</tr>
</thead>
<tbody>
<tr>
<td>sum of interior angles 360°</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>all opposite sides equal in length</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>all opposite angles equal in measure</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>diagonals bisect each other</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>diagonals are perpendicular</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>diagonals equal in length</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>diagonals bisect vertex angles</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Extended Practice Exercises**

29. **ART** The side view of the marble base of a statue is a trapezoid with bases \( AB \) and \( DC \), shown at the right. Prove that \( \angle A \) and \( \angle D \) are supplementary. (Hint: Extend \( AD \) to show \( AD \).)

30. What type of figure do you obtain if you join the midpoints of all the sides of an isosceles trapezoid?

**Mixed Review Exercises**

Use the number line below for Exercises 31–36. Find each length. (Lesson 3-1)

31. \( AF \)  
32. \( BE \)  
33. \( DG \)  
34. \( AH \)  
35. \( CH \)  
36. \( DF \)
Choose the word from the list that best completes each statement.

1. When two geometric figures have the same size and shape, they are said to be _____.
2. If a point lies on the ____ of a segment, then the point is equidistant from the endpoints of the segment.
3. A ____ is a quadrilateral with both pairs of opposite sides parallel.
4. A ____ of a polygon is a segment that joins two nonconsecutive vertices.
5. A ____ is a quadrilateral with exactly one pair of parallel sides.
6. A ____ follows directly from a theorem.
7. A ____ of a triangle is a segment where one endpoint is a vertex and the other endpoint is the midpoint of the opposite side.
8. In a ____, all angles have the same measure and all sides have the same length.
9. Two or more lines that intersect at one point are called ____.
10. A polygon where the lines containing the sides have no points in the interior of the polygon is called ____.

### LESSON 4-1 Triangles and Triangle Theorems, p. 150

- The sum of the measures of the angles of a triangle is 180°.
- The measure of an exterior angle of a triangle is equal to the sum of the measures of the two nonadjacent (remote) interior angles.

Find the value of \( x \) in each figure.

11. [Diagram: \( x \) 144° \( x \)]
12. [Diagram: \( x \) \( 2x - 20 \) \( 50 \)°]
13. [Diagram: \( (4x + 20) \) 101° \( x \)]
14. [Diagram: \( x \) (2\( x + 21 \))°]
15. [Diagram: \( x \) \( 80 \)° \( 3x - 22 \)°]
16. [Diagram: \( x \) \( 2x \) \( (x - 20) \)°]
LESSON 4-2  Congruent Triangles, p. 154

Three postulates for proving that two triangles are congruent are the SSS (Side-Side-Side) Postulate, the SAS (Side-Angle-Side) Postulate, and the ASA (Angle-Side-Angle) Postulate.

In each case, name a pair of congruent triangles. Then name the postulate you could use to prove the triangles congruent. You do not need to write a proof.

17. 18.

LESSON 4-3  Congruent Triangles and Proofs, p. 160

Base angles of an isosceles triangle are congruent.

Find the value of \( x \) in each figure.

21. 22. 23.

LESSON 4-4  Altitudes, Medians, and Perpendicular Bisectors, p. 164

An altitude of a triangle is the perpendicular segment from a vertex to the line containing the opposite side. A median of a triangle is a segment whose endpoints are on a vertex of the triangle and the midpoint of the opposite side.

For Exercises 24–25, use the figure at the right.

24. Name a median of \( \triangle ABC \).
25. Name an altitude of \( \triangle ABC \).
26. Draw an obtuse triangle. Sketch all the altitudes and the medians.

LESSON 4-5  Problem Solving Skills: Write an Indirect Proof, p. 170

To write an indirect proof, the first step is to assume temporarily that the conclusion is false.

Write Step 1 of an indirect proof of each statement.

27. If a triangle is obtuse, then it cannot have a right angle.
28. If two parallel lines are cut by a transversal, then alternate exterior angles are congruent.
29. The angle bisector of the vertex angle of an isosceles triangle is also an altitude of the triangle.
LESSON 4-6 ■ Inequalities in Triangles, p. 172

- The sum of the lengths of two sides in a triangle is greater than the length of the third side.

Determine if the given measures can be lengths of the sides of a triangle.

30. 19 cm, 10 cm, 8 cm
31. 6 ft, 7 ft, 13 ft
32. 8 m, 8m, 15 m

LESSON 4-7 ■ Polygons and Angles, p. 178

- The sum of the measures of the angles of a convex polygon with $n$ sides is $(n - 2)180^\circ$.

Find the unknown angle measure or measures in each figure.

33.  34.  35.

36. Find the sum of the measures of the angles of a polygon with 11 sides.

LESSON 4-8 ■ Special Quadrilaterals: Parallelograms, p. 182

- If a figure is a parallelogram, the opposite sides are equal in length, the opposite angles are equal in measure, and the diagonals bisect each other.

In the figure $OE = 19$ and $EU = 12$. Find each measure.

37. $LE$
38. $OJ$
39. $m\angle OJL$
40. $OU$
41. $OL$
42. $JU$

LESSON 4-9 ■ Special Quadrilaterals: Trapezoids, p. 188

- The length of the median of a trapezoid equals half the sum of the lengths of the bases.

A trapezoid and its median are shown. Find the value of $a$.

43. 14 in.
44. $a$
45. 26 m

CHAPTER INVESTIGATION

EXTENSION Write a report about the design and model of your truss bridge. Include an explanation as to why an actual bridge constructed from your model would support the necessary weight.
Chapter 4 Assessment

Find the value of \( x \) in each figure.

1. \( \angle \) \( \frac{3x - 8}{5x - 1} \)°

2. \( \angle \) \( (2x - 30)° \)

3. \( \angle \) \( 74° \)

4. \( \angle \) \( x° \)

5. \( \angle \) \( 40° \)

6. \( \angle \) \( x° \)

LETR is a trapezoid. 
CD is a median. 

JETW is a parallelogram.

Complete the congruence statement. Name the postulate you could use to prove the triangles congruent. (You do not need to write a proof.)

7. \( \triangle ARM \approx \triangle TRM \)

8. \( \triangle LED \approx \triangle OED \)

9. Write a two-column proof.
   \( \text{Given } TR \cong RE, WT \cong WE \)
   \( \text{Prove: } \triangle TRW \cong \triangle ERW \)

10. Draw an acute triangle and sketch the perpendicular bisectors of all the sides.

11. Suppose you are asked to write an indirect proof of the following statement: If a triangle is equilateral, then it cannot have two sides of unequal lengths. Write Step 1 of the indirect proof.

12. Can a triangle have sides that measure 45 mm, 19 mm, and 23 mm? Explain.

13. In the figure at the right, give a range of possible values for \( x \).

14. Find the sum of the measures of the angles of a polygon with 13 sides.

15. A figure and the result of the first two iterations are shown. Show the result of the third iteration.

mathmatters3.com/chapter_assessment
7. If the following statement is to be proved using indirect proof, what assumption should you make at the beginning of the proof? (Lesson 4-5)

If two sides of a triangle are not congruent, then the angles opposite those sides are not congruent.

A) If two sides of a triangle are congruent, then the angles opposite those sides are congruent.
B) If two sides of a triangle are congruent, then the angles opposite those sides are not congruent.
C) If two sides of a triangle are not congruent, then the angles opposite those sides are congruent.
D) If two angles of a triangle are congruent, then the sides opposite those angles are congruent.

8. Determine which set of numbers can be lengths of the sides of a triangle. (Lesson 4-6)

A) 5 m, 10 m, 20 m  B) 9 in., 10 in., 14 in.
C) 1 km, 2 km, 3 km  D) 8 ft, 15 ft, 29 ft

9. The figure below is a parallelogram with diagonals. Which statement is not true? (Lesson 4-8)

A) VZ = VX  B) WX = ZY
C) WZ || XY  D) WY = XZ

Test-Taking Tip

Question 5

Read the question carefully to check that you answered the question that was asked. In Question 5, you are asked to find the measure of ∠POM, not the value of x or the measure of ∠MOD.
Part 2  Short Response/Grid In

Record your answers on the answer sheet provided by your teacher or on a sheet of paper.

10. Refer to the diagram below to find the number of elements in $C'$. (Lesson 1-3)

11. On Mercury, the temperatures range from $805^\circ F$ during the day to $-275^\circ F$ at night. Find the difference between these temperatures. (Lesson 1-4)

12. The bee hummingbird of Cuba is $\frac{1}{4}$ the length of the giant hummingbird. If the length of the giant hummingbird is $8\frac{1}{4}$ in., find the length of the bee hummingbird. (Lesson 1-5)

13. At the beginning of each week, Lina increases the time of her daily jog. If she continues her pattern, how many minutes will she spend jogging each day during her fifth week of jogging? (Lesson 2-1)

<table>
<thead>
<tr>
<th>Week</th>
<th>Time Jogging</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8 min</td>
</tr>
<tr>
<td>2</td>
<td>16 min</td>
</tr>
<tr>
<td>3</td>
<td>24 min</td>
</tr>
<tr>
<td>4</td>
<td>32 min</td>
</tr>
</tbody>
</table>

14. What is the ordered pair for the point in the graph below? (Lesson 2-2)

15. The following are Tom’s test scores. What is the mean of the data? (Lesson 2-7)

16. In the figure, $AB = 45$. Find $AQ$. (Lesson 3-1)

17. In the figure, $RS \parallel PY$. Find $m \angle RPY$. (Lesson 3-4)

18. The measure of one acute angle of a right triangle is $63^\circ$. What is the measure of the other acute angle? (Lesson 4-1)

19. Find the value of $x$ in the figure. (Lesson 4-3)

20. If a convex polygon has 8 sides, find the sum of the interior angles. (Lesson 4-7)

Part 3  Extended Response

Record your answers on a sheet of paper. Show your work.

21. Write a two column proof. (Lesson 4-2)

Given $\overline{FB}$ is a perpendicular bisector of $AC$.

Prove $\triangle AFB \cong \triangle CFB$

22. Two segments with lengths 3 ft and 5 ft form two sides of a triangle. Draw a number line that shows possible lengths of the third side. Explain your reasoning. (Lesson 4-6)